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OPTIMAL SYNTHESIS IN CERTAIN TERMINAL CONTROL PROBLEMS

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A class of problems of terminal control of multi-dimensional systems reducible to one-dimensional is investigated. Similar problems of time-optimal response for autonomous and nonautonomous systems were considered, for instance, in [1]. The solution derived here is based on sufficient conditions of the dynamic programming method [2]. Ways are developed for the derivation of an analytic solution of the problem of synthesis and, also, for the determination of optimal-phase trajectory and of programmed control. Problems of energy-optimal consumption for stabilizing the rotations of a dynamically symmetric solid body using a limited power motive system are solved [3].

1. Statement of the problem. We consider the controlled system

$$\dot{x} = f(t, x) + b(t, h) S(t, x)u, \quad h = |x|, \quad x(t_0) = x_0 \tag{1.1}$$

where x is the n-vector of the phase state whose values are contained in some neighborhood of point $x = 0$ which includes the initial point x_0; $t \in [t_0, T]$ is the time and T a specified quantity; u is the n-vector of control, $b \geqslant b_0 > 0$ is a scalar, S is an orthogonal $(n \times n)$-matrix: $S^{-1} = S'$. Functions f, b and S are assumed to be fairly smooth and such that the substitution of the admissible continuous control $u(t)$ which at instant $t = T$ bring the phase point to the coordinate origin $x = 0$ yields the unique solution $x(t, [u])$ $(x(T, [u]) = 0)$ for system (1.1). It is further assumed that the vector function $f(t, x)$ has the property [1]

$$\eta' f(t, x) = a(t, h), \quad \eta = x / h \tag{1.2}$$

We pose the following problem of optimal control of system (1.1):

$$x(T) = 0, \quad J[u] = \int_{t_0}^{T} u^2 \, dt \to \min_u \tag{1.3}$$

No additional constraints are imposed at this stage on the control function u. The physical meaning of the criterion of performance of control (1.3) may be defined as the minimization of energy consumed by the control, in which case the quantity u^2 represents the expended power [3].

We have to determine the optimal control in the form of synthesis of $u(t, x)$ which at the specified instant of time $t = T$ brings the phase point of system (1.1) to the coordinate origin $x(T) = 0$, provides the optimal phase trajectory $x(t, t_0, x_0)$, and minimizes functional J (1.3).

Solution of the stated problem is constructed using the sufficient conditions of optimality of the dynamic programming method [2]. Applying the device similar to that set forth in [1] it is possible to show that the solution of problem (1.1), (1.3) with condition (1.2) consists of solving some terminal control problem for the scalar variable h. Thus, by multiplying system (1.1) on the left by η' we obtain the equation

$$h^{\cdot} = a(t, h) + b(t, h)\eta' v, \quad h(t_0) = h_0, \quad h(T) = 0 \qquad (1.4)$$

where v is the new control

$$v = S(t, x)u \quad (u = S'v), \quad J[u] = J[v] = \int_{t_0}^{T} v^2\, dt \to \min_v \qquad (1.5)$$

It follows from (1.4) and (1.5) that $J \to \min_v$ when $v = w\eta$, where w is the unknown scalar control such that $J[w] \to \min_w$. As the result, we obtain the terminal control problem

$$h^{\cdot} = a(t, h) + b(t, h)w, \quad h(t_0) = h_0 \qquad (1.6)$$

$$h(T) = 0, \quad J[w] = \int_{t_0}^{T} w^2\, dt \to \min_w$$

which by using the dynamic programming method is reduced to solving the Cauchy problem for Bellman's function $V(t, h)$

$$\partial V / \partial t + a(t, h)\, \partial V / \partial h - {}^1/_4 b^2(t, h)(\partial V / \partial h)^2 = 0 \qquad (1.7)$$

$$w(t, h) = -{}^1/_2\, b(t, h)\partial V / \partial h, \quad V(T, h(T)) = V(T, 0) = 0$$

The Cauchy problem for the input control problem (1.1) – (1.3) reduces to an equation of the form (1.7), since the equation and the boundary condition for Bellman's function $W(t, x)$ of the input control problem (1.1) – (1.3)

$$\partial W / \partial t + \partial W / \partial x\, f(t, x) - {}^1/_4 b^2(t, h)(\partial W / \partial x)^2 = 0 \qquad (1.8)$$

$$v'(t, x) = -{}^1/_2 b(t, h)\partial W / \partial x, \quad W(T, x(T)) = W(T, 0) = 0$$

imply that

$$W(t, x) = V(t, h), \quad \partial W / \partial x = \partial V / \partial h \eta' \qquad (1.9)$$

$$v(t, x) = -{}^1/_2 b(t, h)\, \partial V / \partial h \eta = w\eta$$

The substitution of (1.9) into (1.8) yields problem (1.7), hence a solution of (1.7) also satisfies (1.8) and in accordance with (1.5) and (1.9) it synthesizes the solution of the control problem (1.1) – (1.3).

2. Derivation of the optimal synthesis. It is not generally possible to obtain a solution for the control problem (1.6) or for the Cauchy problem in the case of arbitrary functions $a(t, h)$ and $b(t, h)$. The following algorithm for solving the problem of synthesis, based on the necessary conditions of the maximum principle, is proposed on the assumption that these two functions are piecewise differentiable with respect to h, $h \in [0, h_0]$ and continuous with respect to t, $t \in [t_0, T]$. When applied to problem (1.6) these conditions are of the form

$$H = -w^2 + (a + bw)p \to \max_w \qquad (2.1)$$

$$h^{\cdot} = a(t, h) + b(t, h)w, \quad h(t_0) = h_0 \quad h(T) = 0$$

$$p^{\cdot} = -p(\partial a / \partial h + \partial b / \partial h\, w), \quad p(T) = p_T$$

The maximum value H^* of Hamiltonian H is

$$H^* = pa\,(t,\,h) + {}^1/_4 p^2 b^2(t,\,h), \quad w^* = {}^1/_2 pb\,(t,\,h) \tag{2.2}$$

and the equality

$$H^*(t) = H^*(T) - \int_t^T \frac{\partial H}{\partial t}\bigg|^* dt', \quad \frac{\partial H}{\partial t} = p\frac{\partial a}{\partial t} + \frac{1}{2}p^2 b\frac{\partial b}{\partial t} \tag{2.3}$$

holds for the solution of system (2.1) [1].

It is assumed below that h and the conjugate variable p are known from the solution of system (2.1) for $w = w^*$

$$h^* = h\,(t,\ t_0,\ h_0), \quad p^* = p\,(t,\ t_0,\ h_0) \tag{2.4}$$

The programmed and positional controls w_p and w_s, respectively, determined in conformity with (2.2), are

$$w_p(t,\ t_0,\ h_0) = {}^1/_2 p\,(t,\ t_0,\ h_0)\ b\,(t,\ h\,(t,\ t_0,\ h_0)) \tag{2.5}$$

$$w_s\,(t,\ h) = w_p\,(t,\ t,\ h) = {}^1/_2\ p\,(t,\ t,\ h)\ b\,(t,\ h)$$

Minimum value of functional J^* and the Bellman function V are similarly determined

$$J^* = V\,(t_0, h_0) = \int_{t_0}^T w_p^2\,(t, t_0, h_0)\,dt, \quad V\,(t, h) = \int_t^T w_p^2\,(\tau, t, h)\,d\tau \tag{2.6}$$

The substitution of optimal control $u^* = w^* S'\eta$ into (1.1) yields the Cauchy problem in ordinary differential equations whose analytic solution can be obtained in a number of cases on the basis of the general solution of system (1.1) with $u \equiv 0$.

Certain particular solutions of problem (1.1) – (1.3) are presented below.

1). Let $a = \alpha\,(t)$, and $b = \beta\,(t)$; then $p = \text{const}$, and the optimal control is determined in conformity with (2.5) as the program and the synthesis

$$w_p(t,\ t_0,\ h_0) = {}^1/_2\beta\,(t)p\,(t_0,\ h_0), \quad w_s\,(t,\ h) = {}^1/_2\beta\,(t)p\,(t,\ h) \tag{2.7}$$

$$p^* = p\,(t_0, h_0) = -2\left[h_0 + \int_{t_0}^T \alpha\,(t)\,dt\right] \Big/ \int_{t_0}^T \beta^2\,(t)\,dt$$

The minimal value of functional J^* and the Bellman function V, determined with the use of (2.6) and allowance for (2.7), are

$$J^* = V\,(t_0, h_0) = \frac{p^{*2}}{4}\int_{t_0}^T \beta^2\,(t)\,dt, \quad V\,(t, h) = \frac{1}{4}\,p^2\,(t, h)\int_t^T \beta^2\,(\tau)\,d\tau \tag{2.8}$$

Direct differentiation shows that function V is the solution of (1.7) and (1.8). The substitution of $w_s\,(t,\,h)$ defined by (2.7) into (2.1) yields a linear equation of motion for h with feedback. Its solution is obtained in the form of quadrature

$$h\,(t, t_0, h_0) = \left[h_0 + \int_{t_0}^T \alpha\,(t)\,dt\right]\int_t^T \beta^2\,(\tau)\,d\tau \Big/ \int_{t_0}^T \beta^2\,(t)\,dt - \int_t^T \alpha\,(\tau)\,d\tau \tag{2.9}$$

It follows from (2.9) that for any t_0, T $(t_0 < T)$, h_0 we have $h(T, t_0, h_0) = 0$.

After the substitution of the expression $u^* = w^* S'\eta$ into (1.1) the construction of the optimal phase trajectory $x(t, t_0, x_0)$ reduces to solving the Cauchy problem. Let $f(t, x) = g(x) + \alpha(t)\eta$, where α is a scalar and $\eta' g(x) \equiv 0$, i.e. g is the vector of gyroscopic forces [1] and $g(x)$ is a homogeneous function of x of power $m \geqslant 1$: $g(kx) = k^m g(x)$. The system (1.1) for the phase vector x with feedback

$$x^{\cdot} = g(x) + \alpha(t)\eta - \beta^2(t)\eta\left[h + \int_t^T \alpha(\tau)\,d\tau\right] \Big/ \int_t^T \beta^2(\tau)\,d\tau, \quad x(t_0) = x_0 \qquad (2.10)$$

is reduced by the substitution $x = hz$ to the form of a system with invariant norm [1]

$$\frac{dz}{ds} = g(z), \quad s = \int_{t_0}^{t} h^{m-1}(\tau, t_0, h_0)\,d\tau, \quad z(0) = z_0 = \eta_0, \quad |z| = 1 \qquad (2.11)$$

If the general solution of system (2.10) is known for $\alpha = \beta \equiv 0$: $x^{\cdot} = g(x)$, $x(t_0) = x_0$ and is given in form $x = \varphi(t - t_0, c, h_0)$, where c may, for instance, be the vector of directional cosines, $c = \eta_0$, the solutions of Eqs. (2.11) and (2.10) are, respectively, of the form

$$z = \varphi(s, z_0, 1), \; x = h(t, t_0, h_0)\,\varphi(s, \eta_0, 1)$$

The phase trajectory in the case of $f(t, x) = \delta(t)\,g(x) + \alpha(t)\,\eta$, where $\delta(t)$ is a scalar function, is determined in a similar manner. In that case δh^{m-1} appears in the integrand of the expression for s in (2.11).

2). Let now $a(t, h) = \gamma(t)h + \alpha(t)$ and $b(t, h) = \beta(t)$; the optimal program w_p and synthesis w_s are then of the form

$$w_p(t, t_0, h_0) = {}^1/_2\beta(t)\,p_T(t_0, h_0)\,\Gamma(T, t) \qquad (2.12)$$

$$w_s(t, h) = {}^1/_2\beta(t)\,p_T(t, h)\,\Gamma(T, t)$$

$$p_T{}^* = p_T(t_0, h_0) = -2\,[h_0\Gamma(T, t_0) + \mathrm{A}(T, t_0)]\,\mathrm{B}(T, t_0)$$

$$\Gamma(T, t) = \exp\left[\int_t^T \gamma(\tau)\,d\tau\right], \quad \mathrm{A}(T, t_0) = \int_{t_0}^T \alpha(t)\,\Gamma(T, t)\,dt$$

$$\mathrm{B}(T, t_0) = \int_{t_0}^T \beta^2(t)\,\Gamma^2(T, t)\,dt$$

The minimal value of functional J (1.3) is then determined on the basis of the first of formulas (2.6)

$$J^* = V(t_0, h_0) = {}^1/_4\,p_T{}^2(t_0, h_0),\; B(T, t_0) \qquad (2.13)$$

In conformity with (2.6) Bellman's function V is

$$V(t, h) = [h\Gamma(T, t) + \mathrm{A}(T, t)]^2 \,/\, \mathrm{B}(T, t) \qquad (2.14)$$

The substitution of expression (2.14) into (1.7) and (1.8) shows that V is the sought Bellman's function. The absolute value of vector x is a function of h when

$w^* = w_s(t, h)$ and in conformity with Eq. (2.1) decreases with negative linear feedback whose coefficient increases indefinitely when $t \to T$. The solution of that equation is of the form

$$h(t, t_0, h_0) = h_0 [\Gamma(t, t_0) - \Gamma(T, t)\Gamma(T, t_0) \mathrm{B}(t, t_0) / \mathrm{B}(T, t_0)] + \mathrm{A}(t, t_0) - \Gamma(T, t) \mathrm{A}(T, t_0) \mathrm{B}(t, t_0) / \mathrm{B}(T, t_0) \qquad (2.15)$$

which implies that $h(T, t_0, h_0) = 0$ for any t_0, T, and h_0.

The optimal phase trajectory $x(t, t_0, x_0)$ is obtained by integrating a Cauchy problem similar to (2.10) after substitution of the expression $u^* = w^* S' \eta$ into (1.1). In investigations of certain applied problems (see Sect. 4) function f is of the form $f(t, x) = \delta(t) g(x) + \gamma(t) x + \alpha(t) \eta$ (δ, γ, and α are scalar functions), where g is the gyroscopic vector. The optimal trajectory

$$x = hz, \quad z = \varphi(s, \eta_0, 1), \quad s = \int_{t_0}^{t} \delta(\tau) h^{m-1}(\tau, t_0, h_0)\, d\tau$$

is derived in a manner similar to that used in Sect. 1 above. Then, on the basis of the expression for the vector function $x(t, t_0, x_0)$, we obtain the control u^* in the form of a program.

3). When $a = \alpha(h)$ and $b = \beta(h)$, where α and β are fairly smooth, for instance continuously differentiable functions h, $h \in [0, h_0]$, the solution of problem (1.6) is derived as follows. Since according to (2.3) the Hamiltonian H^* (2.2)

$$H^*(t) = H^*(T) = {}^1/_4 p_T^2 \beta^2(0) + p_T \alpha(0)$$

is constant, hence for p we have the expression

$$p = p(h, p_T) = -[2 / \beta^2(h)]\{\alpha(h) \mp [\alpha^2(h) + H^*(T)\beta^2(h)]^{1/2}\} \qquad (2.16)$$

The substitution of (2.16) into w^* and then into (2.1) yields for h the equation with separating variables

$$\pm \int_{h_0}^{h} \frac{dl}{[\alpha^2(l) + H^*(T)\beta^2(l)]^{1/2}} = t - t_0 \qquad (2.17)$$

which shows that the plus sign is to be taken in formula (2.16) for p.

Since $h(T) = 0$, p_T is the root of Eq. (2.17) when $t = T$ and $h = 0$. We substitute the obtained expression $p_T^* = p_T(T - t_0, h_0)$ into (2.16) and, as the result, obtain the control w^* in the form of synthesis $w_s(T - t, h) = {}^1/_2 \beta(h) p(h, p_T(T - t, h))$. The implicit function $h(t - t_0, T - t_0, h_0)$ is defined by the quadrature (2.17) after the substitution $p_T^* = p_T(T - t_0, h_0)$. Using formulas (2.5) and (2.6), and the previously described scheme it is possible to derive the optimal control $w_p(t - t_0, T - t_0, h_0)$, the minimum value of functional J^* and the Bellman function V, and the optimal phase trajectory x. If $f(t, x) = \delta(t) g(x)$, i.e. $\alpha \equiv 0$, then $p = p_T \beta(0) / \beta(h)$, where

$$p_T = -\frac{2}{\beta(0)(T - t_0)} \int_0^{h_0} \frac{dh}{\beta(h)} = -\frac{2}{\beta(0)(T - t)} \int_0^{h} \frac{dl}{\beta(l)}$$

The synthesis of control w_s and the formula for h are of the form

$$w_s(T-t,h) = -\frac{1}{T-t}\int_0^h \frac{dl}{\beta(l)}, \quad \int_{h_0}^h \frac{dl}{\beta(l)} = -\frac{t-t_0}{T-t_0}\int_0^{h_0}\frac{dh}{\beta(h)}$$

and the phase trajectory x is defined by the system of equations with feedback

$$x^\cdot = \delta(t)\, g(x) - \frac{x}{h}\frac{\beta(h)}{T-t}\int_0^h \frac{dl}{\beta(l)}, \quad x(t_0) = x_0$$

The substitution of $x = zh$ with $|z| = 1$ yields a Cauchy problem for equations with invariant norm (2.11). The order of that system may be lowered by two, and in a number of applied problems its integration can be carried out to the end (see Sect. 4).

3. Generalization of the problem of terminal control. The multi-dimensional optimal control problem can be reduced to a one-dimensional of the type (1.6) in a more general case. Let, for example, the system of equations of motion be of the form

$$x^\cdot = f(t, x, |u|) + b(t, h, |u|)\, S(t, x, u)u, \quad x(t_0) = x_0 \quad (3.1)$$

where f, b, and S are analogous to those considered above.

We have the problem of bringing the phase point of system x from the initial state $x(t_0) = x_0$ on the manifold

$$|x(T)| \leqslant M \quad (|x_0| > M), \quad |x(T)| \geqslant M \quad (|x_0| < M) \quad (3.2)$$

in such a way that the functional

$$J = F(h(T)) + \int_{t_0}^T G(t, h, |u|)\, dt, \quad h = |x| \quad (3.3)$$

attains the lowest possible value. The control vector u may be subjected to the supplementary constraint [1]

$$|u| \leqslant u_0, \quad |u| = (u_1^2 + \ldots + u_n^2)^{1/2} \quad (3.4)$$

Then, using the reasoning of Sect. 1, we obtain the equivalent control problem

$$h^\cdot = a(t, h, |w|) + b(t, h, |w|)\, w, \quad h(t_0) = h_0 \quad (3.5)$$

$$h(T) \lesseqgtr M \quad (h_0 \gtreqless M), \quad J = F(h(T)) + \int_{t_0}^T G(t, h, |w|)\, dt \to \min_{|w| \leqslant u_0}$$

The Bellman equation of the input problem (3.1) −(3.4) of optimal control, owing to the central symmetry of formulas (3.5) with allowance for conditions of the type (1.2) is thus reduced to the form (1.7)

$$\frac{\partial V}{\partial t} + \min_{|w| \leqslant u_0}\left\{\frac{\partial V}{\partial h}[a(t,h,|w|) + b(t,h,|w|)w] - G(t,h,|w|)\right\} = 0, \quad (3.6)$$
$$V(T, h(T)) = F(h(T)), \quad h(T) \lesseqgtr M$$

Let us assume that the terminal control problem (3.5), (3.6) has been solved and that the optimal control has been determined in the form of program $w_p(t, t_0, h_0)$ or by the feedback $w_s(t, h)$, using methods of Sect. 2. For the determination of the attitude control $u_s(t, x)$ we then obtain the final control $S(t, x, u)u = w_s(t, h)\eta$, $|u| \leqslant u_0$, from which we have function $u^* = u_s(t, x)$. If matrix S depends only on $|u|$ or is altogether independent of u, then $u_s(t, x) = w_s S'(t, x, |w_s|)\eta$. Construction of optimal phase trajectory is carried out in the manner described in Sect. 2, on the assumption that $f(t, x, |u|) = \chi(t, |u|)g(x) + a(t, h, |u|)\eta$.

Note that the idealization, as expressed by relations (3.1) – (3.4) is not always applicable to practical problems. For instance, relationships of the type (1.2) are, as a rule, satisfied with some error

$$\eta' f(t, x, u) = a(t, h, |u|) + \varepsilon\varphi(t, x, u)$$

in which $\varepsilon \in [0, \varepsilon_0]$ is some small quantity and $\varphi(t, x, u)$ is a bounded function for $t \in [t_0, T]$, $|x| \leqslant |x_0|$, $|u| \leqslant u_0$. Other perturbing factors may also affect system (3.1). When the indicated idealization is inapplicable, it becomes necessary either to estimate such perturbations or derive an approximate solution for the terminal control problem taking into account the small parameter ε.

The problem of control was investigated by methods of the theory of perturbations in [4 – 6]. For autonomous systems of the type (1.1) the author has developed in [7, 8] a method of derivation of an approximate solution for problems of time-optimal response with constraint on control (3.4), which is based on the sufficient conditions of optimality [2].

4. Optimal control of motion of a solid body relative to its center of mass. We consider the problem of control of rotations of a dynamically symmetric body in Euler's case [1, 7, 8]

$$I\omega_1^{\cdot} + (I_3 - I)\omega_2\omega_3 = M_1, \quad \omega_1(t_0) = \omega_{10} \tag{4.1}$$

$$I\omega_2^{\cdot} - (I_3 - I)\omega_1\omega_3 = M_2, \quad \omega_2(t_0) = \omega_{20}$$

$$I_3\omega_3^{\cdot} = M_3, \quad I, I_3 = \text{const}, \quad \omega_3(t_0) = \omega_{30}$$

where I and I_3 are the principal central moments of inertia of the body, M_i are components of the external moment of forces relative to each of the attached axes, and t_0, ω_{i0} are input data $(i = 1, 2, 3)$.

1). We begin by considering a control scheme of the kind shown in Fig. 1, where (f_3, f_3) is a pair of fixed motors that produce a moment of forces about the axis of symmetry $O\omega_3$ and $(f_\perp, f_\perp)$ is a pair of vernier motors on that axis, which generate the moment of control forces about the axes $O\omega_1$ and $O\omega_2$. In the absence of other effects we have

$$M_1 = 2lf_\perp \sin\psi, \quad M_2 = 2lf_\perp \cos\psi, \quad M_3 = 2rf_3 \tag{4.2}$$

$$0 \leqslant f_\perp \leqslant f_{\perp 0}, \quad 0 \leqslant \psi \leqslant 2\pi, \quad 0 \leqslant f_3 \leqslant f_{30}$$

where $2l$ and r are the respective linear characteristics of the system.

It is assumed that the angular velocity ω_3 rotation of the body about the dynamic axis of symmetry varies in conformity with the selected control law $u_3(t)$

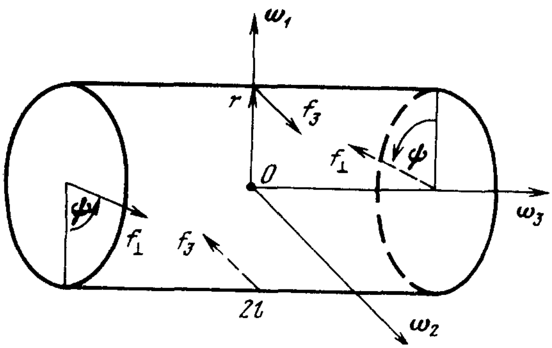

Fig. 1

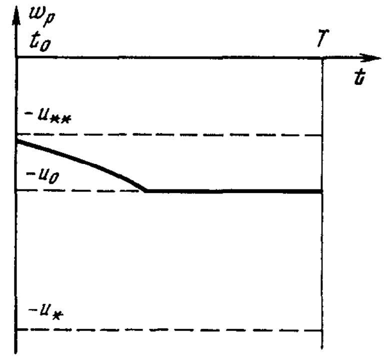

Fig. 2

which, for example, at the specified instant of time $t = T$ brings that velocity to the required value ω_{3T}.

Let us consider the problem of optimal extinguishing of vector (ω_1, ω_2)

$$\omega_3(t) = \omega_{30} + \int_{t_0}^{t} u_3(\tau)\, d\tau, \quad \omega_{1,2}(T) = 0, \quad J = \int_{t_0}^{T} (u_1^2 + u_2^2)\, dt \to \min \qquad (4.3)$$
$$u_3 = M_3 I_3^{-1}, \quad u_{1,2} = M_{1,2} I^{-1}, \quad (u_1^2 + u_2^2)^{1/2} \leqslant u_0 \quad (u_0 = 2lf_{\perp 0} I^{-1})$$

If the control force $f_\perp$ in (4.2) is created by a motive power system of limited power [3], the functional J in (4.3) has the meaning of energy used by the control. The equations of motion (4.1) for ω_1 and ω_2 are reduced to the form

$$\omega_1^{\cdot} = -\nu(t)\omega_2 + u_1, \quad \omega_2^{\cdot} = \nu(t)\omega_1 + u_2 \qquad (4.4)$$
$$\nu(t) = (I_3 - I)I^{-1}\omega_3(t)$$

System (4.4) satisfies condition (1.2), and the solution of problem of control with condition $u_0 \geqslant \omega_{\perp 0}(T - t_0)^{-1}$ is of the form

$$V(t, \omega_\perp) = \omega_\perp^2 (T - t)^{-1}, \; w^* = -\omega_\perp (T - t)^{-1}, \; \omega_\perp = (\omega_1^2 + \omega_2^2)^{1/2} \qquad (4.5)$$
$$u_{1,2}^* = -\omega_{1,2}(T - t)^{-1}$$

The optimal phase trajectory and the minimum value of the functional are obtained using methods of Sect. 2. They are

$$\omega_1^* = (T - t)(T - t_0)^{-1}(\omega_{10}\cos s - \omega_{20}\sin s) \qquad (4.6)$$
$$\omega_2^* = (T - t)(T - t_0)^{-1}(\omega_{10}\sin s + \omega_{20}\cos s), \qquad s = \int_{t_0}^{t} \nu(\tau)\, d\tau$$
$$J^* = \frac{\omega_{\perp 0}^2}{T - t_0}, \quad \omega_\perp = \omega_{\perp 0}(T - t)(T - t_0)^{-1}$$

When $u_0 < \omega_{\perp 0}(T - t_0)^{-1}$ the stated problem has no solution. If, however, constraint (4.3) on u_1 and u_2 is virtually absent, i.e. $u_0 \to \infty$, then (4.5) and

(4.6) imply that when $t_0 \to T$, the programmed controls become delta functions [9], while the integral functional becomes divergent: $J^* \to +\infty$, and conversely, when $T - t_0 \to \infty$, $J^* \to 0$.

Let us assume that viscous friction forces of the external medium produce a braking moment on the symmetric solid body. Then equations of motion of the type (4.4) assume the form (see 2) in Sect. 2)

$$\omega_1^{\cdot} = -\nu(t)\omega_2 - \gamma\omega_1 + u_1, \quad \omega_2^{\cdot} = \nu(t)\omega_1 - \gamma\omega_2 + u_2 \qquad (4.7)$$

If $\gamma = \text{const}$, the solution of the problem of synthesis is of the form

$$V(t, \omega_\perp) = 2\gamma\omega_\perp^2\Gamma^2(t, T)[1 - \Gamma^2(t, T)]^{-1} \qquad (4.8)$$
$$\Gamma(t, T) = \exp\gamma \quad (t - T)$$
$$u_{1,2} = \omega_{1,2}\omega^*\omega_\perp^{-1}, \; w^* = w_s(t - T, \omega_\perp) = -2\gamma\omega_\perp\Gamma^2(t, T) \times [1 - \Gamma^2(t, T)]^{-1}$$

The optimal phase trajectory, the programmed control, and the minimum of functional for system (4.7) are

$$\omega_1^* = \omega_\perp\omega_{\perp 0}^{-1}(\omega_{10}\cos s - \omega_{20}\sin s), \; \omega_2^* = \omega_\perp\omega_{\perp 0}^{-1}(\omega_{10}\sin s + \omega_{20}\cos s) \qquad (4.9)$$
$$\omega_\perp^* = \omega_{\perp 0}\{\Gamma(t_0, t) - \Gamma(t_0, T)[\Gamma(t, T) - \Gamma(t_0, t)\Gamma(t_0, T)] \times [1 - \Gamma^2(t_0, T)]^{-1}\}$$
$$w^* = w_p(t - T, \; T - t_0, \; \omega_{\perp 0}) = -2\gamma\omega_{\perp 0}\Gamma(t, T)\Gamma(t_0, T) \times [1 - \Gamma^2(t_0, T)]^{-1}$$
$$J^* = 2\gamma\omega_{\perp 0}^2\Gamma^2(t_0, T)[1 - \Gamma^2(t_0, T)]^{-1}, \quad s = \int_{t_0}^{t}\nu(\tau)\,d\tau$$

The validity of the remark that follows formulas (4.6) is confirmed by (4.8) and (4.9). Solution (4.8), (4.9) actually holds if control w_p does not reach the constraint, i.e.

$$u_0 \geqslant u_* = 2\gamma\omega_{\perp 0}\Gamma(t_0, T)[1 - \Gamma^2(t_0, T)]^{-1}$$

if, however,

$$u_0 < u_{**} = \gamma\omega_{\perp 0}\Gamma(t_0, T)[1 - \Gamma(t_0, T)]^{-1} \quad (u_{**} < u_*)$$

then system (4.7) cannot be stabilized within the time interval $T - t_0$. In the intermediate case of $u_{**} < u_0 < u_*$ the programmed control is split in two sections (see Fig. 2)

$$w^* = w_p(t - T, T - t_0, \omega_{\perp 0}) = \begin{cases} -u_0\Gamma(t, t_1), & t_0 \leqslant t \leqslant t_1 \\ -u_0, & t_1 < t \leqslant T \end{cases}$$

where t_1, $t_0 < t_1 < T$ is a certain instant at which control w_p reaches its minimum value u_0. Note that according to (4.9) the optimal control $w_p < 0$ has a tendency of decreasing character for $t \in [t_0, t_1]$. The quantity t_1 is determined by the condition that $\omega_\perp$ vanishes at $t = T$. The current value of $\omega_\perp$ is

$$\omega_{\perp} = \omega_{\perp 0}\Gamma(t_0, t) - u_0\Gamma(0, t) \times \begin{cases} {}^1/_2\,[\Gamma(2t, t_1) - \Gamma(2t_0, t_1)], & t_0 \leqslant t \leqslant t_1 \\ {}^1/_2\,[\Gamma(t_1, 0) - \Gamma(2t_0, t_1)] + \Gamma(t, 0) - \Gamma(t_1, 0), & t_1 < t \leqslant T \end{cases}$$

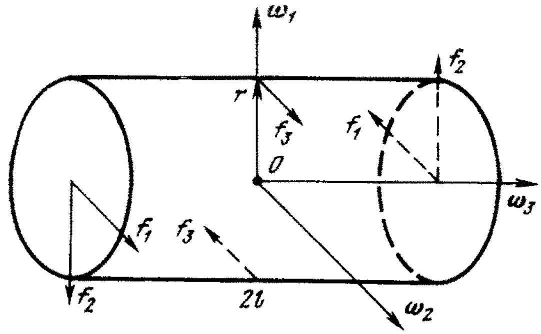

Fig. 3

Solving the related quadratic equation we obtain

$$t_1^*(t_0, \omega_{\perp 0}) = \gamma^{-1}\ln(\Gamma(T, 0) - \gamma\omega_{\perp 0}u_0^{-1}\Gamma(t_0, 0) + \{[\Gamma(T, 0) - \gamma\omega_{\perp 0}u_0^{-1}\Gamma(t_0, 0)]^2 - \Gamma^2(t_0, 0)\}^{1/2})$$

At the limit when $u_0 \downarrow u_{**}$, then $t_1^* \downarrow t_0$, and when $u_0 \uparrow u_*$ then $t_1^* \uparrow T$. In the neighborhood of limit values we have the approximate expressions:

When $u_0 = u_{**}(1 + \varepsilon)$ $(1 \gg \varepsilon > 0)$ we have

$$t_1^* = t_0 + \sqrt{2\varepsilon}\gamma^{-1}[1 - \Gamma(t_0, T)]^{1/2}\Gamma({}^1/_2\,t_0, {}^1/_2\,T) + O(\varepsilon),$$
$$t_1^* - t_0 = O(\sqrt{\varepsilon})$$

and when $u_0 = u_*(1 - \mu)$ $(1 \gg \mu > 0)$

$$t_1^* = T - \mu\gamma^{-1} + O(\mu^2), \quad T - t_1^* = O(\mu)$$

2). Let us now consider a scheme of control by a system of limited total motive power [1,3,7] (see Fig. 3). In that case the equations of motion (4.1) are of the form

$$\begin{aligned} &\omega_1^{\cdot} + \varkappa\omega_2\omega_3 = bu_1, \quad \omega_1(t_0) = \omega_{10}, \quad \varkappa = (I_3 - I)I^{-1} \\ &\omega_2^{\cdot} - \varkappa\omega_1\omega_3 = bu_2, \quad \omega_2(t_0) = \omega_{20}, \quad b = 2l\mu I^{-1} \\ &\omega_3^{\cdot} = b_3u_3, \quad \omega_3(t_0) = \omega_{30}, \quad b_3 = 2r\mu I_3^{-1} \end{aligned} \tag{4.10}$$

where $\mu \approx \text{const}$ is the rate of the working substance consumption under saturation conditions, $u_{1,2,3}$ are the reaction stream discharge velocity $(f_{1,2,3} = \mu u_{1,2,3})$ controllable within wide limits, and the power expended on the control is $N = {}^1/_2\mu(u_1^2 + u_2^2 + u_3^2)$, $N \leqslant N_0$. The minimized functional which represents the system energy E used by the control is

$$E = \frac{\mu}{2}\int_{t_0}^{T}(u_1^2+u_2^2+u_3^2)\,dt, \quad u_1^2+u_2^2+u_3^2 \leqslant u_0^2 = 2\frac{N_0}{\mu} \tag{4.11}$$

We now have the problem of optimal braking the solid body rotations (to bring it to rest) at instant of time $t = T$. Using substitutions

$$x_{1,2} = \omega_{1,2}b^{-1}, \quad x_3 = \omega_3 b_3^{-1}, \quad \chi = \varkappa b_3, \quad E = {}^1/_2\mu J \tag{4.12}$$

we reduce problem (4.10), (4.11) to the form (1.1), (1.3), (3.4). The optimal control on condition that $u_0 \geqslant h_0(T-t_0)^{-1}$ is

$$V(t, h) = h^2(T-t)^{-1}, \quad u_i^* = w^*\eta_i, \quad w^* = -h(T-t)^{-1}$$
$$\eta_i = x_i h^{-1}, \quad h = |x| \quad (i = 1,2,3)$$

It is not possible to stop the rotations of a solid body within the indicated time interval when $u_0 < h_0(T-t_0)^{-1}$. The optimal phase trajectory and the minimum value of functional J are obtained in explicit form

$$x_i^* = hz_i \quad (i = 1,2,3), \quad h = h_0(T-t)(T-t_0)^{-1}$$
$$z_1^* = \eta_{10}\cos s - \eta_{20}\sin s, \quad z_2^* = \eta_{10}\sin s + \eta_{20}\cos s$$
$$z_3^* = \eta_{30}$$
$$s = \chi h_0\eta_{30}(t-t_0)(T-t_0)^{-1}[T - {}^1/_2(t+t_0)], \quad J^* = h_0^2(T-t_0)^{-1}$$

The input quantities ω_i^* and E^* are calculated by inverting formulas (4.12). It should be noted that $s^\cdot = 0$ when $t = T$. The effect of the moment of viscous friction forces is investigated as in Sect. 1.

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